STAT 8025

Lecture 9: Spatio-Temporal Modeling

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- Spatial $\rightarrow$ Spatio-temporal modeling: An important area that itself can be studied in a full course.
- For this lecture, I will give an introduction and list some but not exhaustive resources and references.

Books

- Statistical Analysis of Environmental Space-Time Processes by Le and Zidek (2006)
- Statistics for Spatio-Temporal Data by Cressie and Wikle (2011)
  - You can read together with Spatio-Temporal Statistics with R (https://spacetimewithr.org/index)
- Spatial and Spatio-Temporal Bayesian Models with R-INLA by Blangiardo and Cameletti (2015)
- Spatio-Temporal Methods in Environmental Epidemiology by Shaddick and Zidek (2015)

Spatio-Temporal Modeling

- Treating time as a continuous index $Y(\mathrm{s}; t)$
  - mobile sensing
- Treating time as a discrete index $Y_t(\mathrm{s})$
  - air pollution and meteorological monitoring networks

Discrete Time

Example: A Spatial AR(1)

$$Y_1(\mathrm{s}) = X_1(\mathrm{s})'\boldsymbol{\beta} + w_1(\mathrm{s})$$

$$Y_t(\mathrm{s})|Y_{t-1}(\mathrm{s}) = X_t(\mathrm{s})'\boldsymbol{\beta} + \rho(Y_{t-1}(\mathrm{s}) - X_{t-1}(\mathrm{s})'\boldsymbol{\beta}) + \sqrt{1-\rho^2}w_t(\mathrm{s})$$

where

- $w_t(\cdot)$ iid $\mathcal{GP}(0, C(\cdot,\cdot;\boldsymbol{\theta}))$, say, Matérn with variance 1 and parameters $\boldsymbol{\theta}$
- What is $Cov(Y_1(\mathrm{s}), Y_2(\mathrm{u}))$?
- In genera,

$$Cov(Y_t(\mathrm{s}), Y_{t+d}(\mathrm{u})) = \rho^d C(|\mathrm{s}-\mathrm{u}|)$$

- Limitations?

- The model is stationary: mean, spatial and temporal dependence structures
- The covariance is of the form:

$$C(Y_t(\mathrm{s}), Y_{t'}(\mathrm{u})) = C_T(t, t') \cdot C_S(\mathrm{s}, \mathrm{u})$$

  - Spatial covariance is the same for all times
  - Time autocorrelation is the same for all locations
  - We call this a separable covariance function.
- It is often (if not all) not realistic to assume separability, though it is convenience in modeling and computation
- Test separability? Often done informally, e.g., estimate C_T at different locations; estimate C_S at different times.
- Formal tests are available, for example:
  P. Constantinou, P. Kokoszka, M. Reimherr, Testing separability of space-time functional processes, Biometrika, Volume 104, Issue 2, June 2017, Pages 425-437.

Computation with Separable Covariance

Assume we observe $Y_{(nm)\times 1}$, n locations and m time points. We a separable covariance structure, we have

$$Y \sim N(X\beta, \Sigma_T \otimes \Sigma_S)$$

- $\otimes$ for Kronecker product of matrices
- Properties with Kronecker products:

$$|A_{n\times n} \otimes B_{m\times m}| = |B|^n |A|^m$$

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

- When n and m are both reasonably small, MLE and Bayesian inference will be efficient.
- Prediction can be done using the conditional Gaussian distribution as before.

Discrete Time

- It can be treated as a multivariate time series problem

$$Y_t(s) = X_t(s)'\beta + S(s)'\eta_t + \epsilon_t(s)$$

  - Choose some spatial basis functions $S(\cdot)$ to represent the spatial variation and let the coefficients vary over time
  - Model the coefficients as AR, ARMA etc.
  - Kalman filtering can be used in inference
  - Nonstationary nonseparable spatio-temporal covariance structure
  - See more details in the following papers which are extensions of Fixed rank kriging (FRK) to the spatio-temporal setting:
    - Noel Cressie, Tao Shi & Emily L. Kang (2010) Fixed Rank Filtering for Spatio-Temporal Data, Journal of Computational and Graphical Statistics, 19:3, 724-745.
    - Hai Nguyen, Matthias Katzfuss, Noel Cressie & Amy Braverman (2014) Spatio-Temporal Data Fusion for Very Large Remote Sensing Datasets, Technometrics, 56:2, 174-185.

Other Flexible Models

- Making some components vary $\rightarrow$ easy to specify but not always easy to fit
- Examples:
  - Nonstationary in space but stationary in time
    - We can use a nonstationary $C_S(\cdot)$
    - We can also let ρ vary over space: $Y_t(\mathrm{s}) = \rho(\mathrm{s})Y_{t-1}(\mathrm{s}) + w_t(\mathrm{s})$
  - Nonstationary in time but stationary in space: $Y_t(\mathrm{s}) = \rho_t Y_{t-1}(\mathrm{s}) + \sigma_t w_t(\mathrm{s})$
    - Moving window AR(1) for $\rho(t)$ and σ_t
  - Dynamic linear model:

$$Y_t(\mathrm{s}) = \mathrm{X}_t(\mathrm{s})'\boldsymbol{\beta}_t + \rho(Y_{t-1}(\mathrm{s}) - \mathrm{X}_{t-1}(\mathrm{s})'\boldsymbol{\beta}_{t-1}) + w_t(\mathrm{s})$$

$$\boldsymbol{\beta}_t|\boldsymbol{\beta}_{t-1} \sim N(\mathrm{A}\boldsymbol{\beta}_{t-1}, \Sigma_\beta)$$

- Spatially-varying coefficients model: $Y(\mathrm{s}) = \mathrm{X}(\mathrm{s})'\boldsymbol{\beta}(\mathrm{s}) + \epsilon(\mathrm{s})$

Continuous Time

$$Cov(Y(\mathrm{s}, t_1), Y(\mathrm{u}, t_2)) = C(\mathrm{s}, \mathrm{u}; t_1, t_2)$$

- Now the ‘coordinates’ are from $\mathcal{R}^{d+1}$ instead of $\mathcal{R}^d$
- Stationary: $Cov(Y(\mathrm{s}, t_1), Y(\mathrm{u}, t_2)) = C(\mathrm{s} - \mathrm{u}; t_1 - t_2)$
- Separable: $Cov(Y(\mathrm{s}, t_1), Y(\mathrm{u}, t_2)) = C_S(\mathrm{s}, \mathrm{u})C_T(t_1, t_2)$
- Symmetric: $Cov(Y(\mathrm{s}, t_1), Y(\mathrm{u}, t_2)) = Cov(Y(\mathrm{s}, t_2), Y(\mathrm{u}, t_1))$
- Recall the commonly used covariance functions (Matérn, exponential, etc.). They are symmetric positive definite functions
- Products and sums (and convex combinations) of positive definite functions are still positive definite

Constructing Spatio-temporal Covariance Functions

- Many parametric classes of stationary spatio-temporal covariance functions are constructed as sums or products of those commonly used covariance functions. For example,
  - Separable:
$$C(\mathrm{h}, \mathrm{u}) = C_S(\mathrm{h}) \cdot C_T(u)$$
where $(\mathrm{h}, u) \in \mathcal{R}^d \times \mathcal{R}$ and C_S and C_T are (possibly distinct) parametric classes of isotropic covariance functions as Matérn, powered exponential etc.
- Mixtures of separable covariance functions
  - Examples (De Iaco et al., 2001)
$$C(\mathrm{h}, u) = a_0 C_S^0(\mathrm{h}) C_T^0(u) + a_1 C_S^1(\mathrm{h}) + a_2 C_T^2(u)$$
where $a_i,\ i = 0, 1, 2$ are nonnegative coefficients.

Non-separable Covariance Functions

Bochner's Theorem to consider spectral density
Spatial case: Suppose that C is a continuous and symmetric function of $\mathcal{R}^d$. Then C is a covariance function if and only if

$$C(\mathrm{h}) = \int_{R^d} e^{i(\mathrm{h}'\mathrm{w})} F(d\mathrm{w})$$

where F is a positive finite measure with corresponding spectral density f

$$f(\mathrm{w}) = \frac{1}{(2\pi)^d} \int_{R^d} exp(-i\mathrm{h}'\mathrm{w}) C(\mathrm{h}) d\mathrm{h}$$

- If the spectral density is separable $f(\mathrm{h})f(u)$, then the covariance is separable
- Non-separable spectral densities,
  - Cressie, N. A. & Huang, H. (1999). Classes of nonseparable, spatio-temporal stationary covariance functions. Journal of the American Statistical Association, 94 (448), 1330-1340.
  - Tilmann Gneiting (2002) Nonseparable, Stationary Covariance Functions for Space-Time Data, Journal of the American Statistical Association, 97:458, 590-600.

More reading:

- Dynamical spatio-temporal models (using PDEs) e.g., Wikle and Holan (2011)
  - Deep Integro-Difference Equation Models for Spatio-Temporal Forecasting
- A multi-surrogate higher-order singular value decomposition tensor emulator for spatio-temporal simulators

Summary

- Spatio-tempral modelilng

Preview:

- R examples for spatio-temporal modeling